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Transition delay by surface heating: a zonal analysis for axisymmetric bodies

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It has long been known from linear stability theory that heating a surface immersed in water flow tends to stabilize the boundary layer on the surface, suggesting that there may be a corresponding delay in transition. Experiments confirm the suggestion, but based on intermittency data on a heated body of revolution (Lauchle & Gurney 1984) it has been inferred that incremental changes in transition Reynolds number diminish as the overheat increases. The parameter chosen to locate transition in the analysis leading to this conclusion corresponds to the point where the intermittency is 0.5. However, intermittency distributions in the transition zone on an axisymmetric body may contain what have been called 'subtransitions' (Narasimha 1984). Taking this possibility into account, we formulate here a model for the variation of intermittency with flow Reynolds number at a fixed station on the body, as in the experiments. The rate at which turbulent spots merge with each other is shown to determine the location of subtransition. The transition onset Reynolds number (corresponding to the location where intermittency begins to depart from zero), inferred from the data on the basis of this model, shows a continuing increase with the temperature overheat, a trend in closer agreement with stability theory; but the axisymmetric body geometry results in a very short transition zone, countering in part the benefits of transition delay.

1. Introduction

It has long been realized that surface heating and cooling can exert a significant influence on the stability characteristics of the boundary layer, and hence also on transition location. The sign of the effect depends on the nature of the variation of viscosity with temperature. Thus heating destabilizes the boundary layer in air and stabilizes it in water. Since the viscosity of water is lower when water is hotter the boundary layer has a fuller velocity profile when the surface is heated, which leads to greater stability in a heated water boundary layer. In the case of air, viscosity increases with temperature and heating the wall results in destabilizing the boundary layer. The effect in air was demonstrated in the early experiments of Liepmann & Fila (1947) on a flat plate. The effects are more pronounced in water due to good thermal coupling (higher Prandtl number, $\simeq 7$) and the stronger dependence of viscosity on temperature. The stability calculations of Wazzan, Okamura & Smith (1968) showed that wall heating in water boundary layers could delay transition appreciably. The strong dependence of stability on temperature-dependent variations in viscosity is also evident in the work of Wall & Wilson (1997). The experiments of Barker & Gile (1981), in the entry region of a pipe, confirmed that transition could be delayed but only up to overheats of 8°C: beyond that there was no effect, in contradiction with the calculations of Wazzan *et al.* Lauchle & Gurney (1984, referred to as LG hereinafter) reported experiments on a heated axisymmetric body in a high-speed water tunnel, and inferred that transition Reynolds numbers had gone up from 4.5×10^6 to 3.6×10^7 for an average overheat of 25°C, but there appeared to be a tendency to saturate at large overheats, again contrary to the indications of stability theory.

The use of heating to delay transition can be an attractive option on high-speed under-water vehicles, as waste heat from the propulsion unit that powers the vehicle is usually available at no extra cost. It therefore seems worthwhile to investigate the phenomenon from different points of view.

The LG experiments are particularly interesting as they present intermittency measurements; however, the streamwise distribution was not measured, but only the value of the intermittency γ at a fixed station as the free-stream velocity was varied. The condition at which $\gamma = 0.5$ was taken as an indication of occurrence of transition at the station.

In their analysis of the burst-rate results LG used the model for the transition zone proposed by Dhawan & Narasimha (1958) for flow past a flat plate. However, the transition zone on axisymmetric bodies differs in some important respects from that on two-dimensional bodies. A major reason for this is that, as Rao (1974) pointed out, a turbulent spot will eventually wrap around the axisymmetric body (unless its radius increases too rapidly downstream). This leads to certain peculiarities in the transition zone that have been analysed by Narasimha (1984). Making a specific hypothesis about the shape of the spot as it wraps into a sleeve, this analysis introduced the concept of a 'subtransition' in the transition zone: the intermittency distribution with a subtransition can be very different from that in flat plate flow. In the present work, we show that the location of subtransition can be determined from the transition Reynolds number and the rate of breakdown at the onset of transition.

Our chief objective in this paper is to present a re-analysis of the LG data based on a more appropriate model for intermittency variation with flow Reynolds number at a fixed station on an axisymmetric body, and thence to derive conclusions on the effect of heating on transition delay in water boundary layers. Before we do this, it is necessary to review briefly the basis for the present model for the transition zone in the heated body experiments.

2. The transition zone: a brief review

The transition zone in a boundary layer may be characterized by the intermittency γ, defined as the fraction of time that the flow is turbulent. The intermittency is zero in laminar flow, increases with downstream distance in the transition zone, and tends asymptotically to unity in the fully turbulent zone. This variation is typically due to the formation of turbulent spots (Emmons 1951) and their subsequent growth as they move downstream until they cover the entire boundary layer. In a plane parallel to the surface, spots are typically heart-shaped. Unless there are rapid changes in the external pressure gradient, it is well-established (see e.g. Seifert & Wygnanski 1995) that the spot propagates linearly and that the angle subtended by it at its origin is constant. The head and base of the spot move at constant fractions of the free-stream velocity, so that its length as well as its width at the base grow linearly in time. Narasimha (1957) and Dhawan & Narasimha (1958) showed that the intermittency distribution in two-dimensional flow can be explained by the hypothesis of concentrated break-down, according to which all turbulent spots originate at nearly the same streamwise

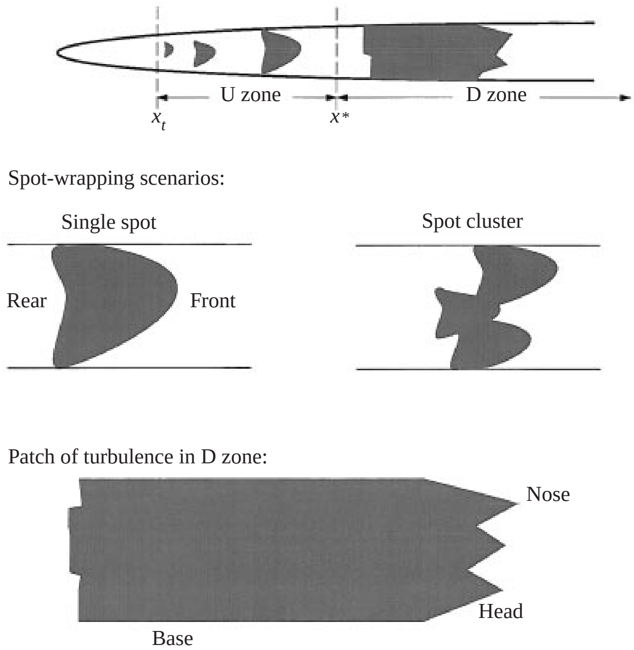

FIGURE 1. Propagation of turbulent spots on an axisymmetric body.

location. The distribution so calculated has received wide experimental support (see e.g. Gostelow & Blunden 1988); there is some indication that at the foot of the intermittency curve, i.e. at very low values of γ, there are slight departures (see e.g. Johnson 1998), but over the bulk of the transition zone the results are entirely adequate.

Comparatively little work, however, has been done on transition zone modelling in axisymmetric boundary layers, in spite of the many possible applications such as in drag estimates for missiles, torpedoes and other underwater bodies. In flow past a flat plate the propagation of the spot is linear, but spot growth characteristics in the axisymmetric case are somewhat different. Consider the flow past an axisymmetric body consisting of a cylinder with a smooth blunt nose whose axis is aligned with the flow (Fig. 1). As pointed out by Rao (1974), a spot that forms at any point on the surface in this flow grows laterally as it moves downstream until it wraps itself around the body. Beyond this point, the spot becomes a 'sleeve', and can grow only in the streamwise direction. Now the primary cause for the growth of the spot is thought to be the destabilization of the laminar boundary layer close to the edges of the spot (Wygnanski, Haritonidis & Kaplan 1979; Glezer, Katz & Wygnanski 1989). For an axisymmetric body, further destabilization is of course not possible in the lateral direction downstream of the location where the spot wraps itself around the body. In the axial direction, however, the spot can continue to destabilize its neighbourhood, and grow as before. If the spot is fully developed and the pressure gradients are not strong, then the propagation velocity of the spot shows no strong variation downstream.

Based on these considerations, Narasimha (1984) proposed the specific hypothesis that the spot, when it forms a sleeve around the body, is best seen as consisting of two parts. The first is a head that extends from the tip of the spot to the first point of contact of the 'wings' as the spot wraps around the body; this part retains its shape and size downstream, and so does not grow. The second is a 'base' extending from the point of contact mentioned above to the trailing edge of the spot; the length of this base grows linearly in time. This idea is illustrated in figure 1. Based on this hypothesis he showed that near the upstream end of the transition zone (hereafter referred to as the U zone), where the spot width is much smaller than the circumference of the body, the intermittency follows a (two-dimensional) law exactly like in flow past a flat plate, and that, downstream of where the spot wraps around (in the D region), the intermittency distribution follows a one-dimensional law. He saw this as a 'subtransition' within the transition zone, as the intermittency distribution changes rapidly from a two- to a one-dimensional law. Evidence for such subtransitions was presented through an analysis of the experimental data of Rao (1974), who measured the streamwise variation of intermittency on a circular cylinder with axis aligned to the flow. It is relevant to emphasize here that the turbulence wrapping itself around the body may follow one of two scenarios: it may be either a single spot (if the spot formation rate at onset is low), or a combination of merged spots or a 'spot cluster' as we shall call it (if the rate is high); in either case there results a continuous patch of turbulence all around the body. Once the circumference is entirely covered by turbulence, the patch of turbulence can grow only in the streamwise coordinate, and subtransition may be said to have taken place.

LG have reported extensive data on intermittency on an axisymmetric body, with and without surface heating. Their data however refer to the variation of intermittency at a fixed station on the body as the free-stream velocity is varied. To interpret these data, we need to supplement the available theory for streamwise intermittency distributions to take into account the changes in onset location and transition zone length that will inevitably accompany changes in free-stream velocity. Fortunately there is in fact now a body of results (Narasimha 1985) that enable us to do this, as we shall show here. Once this is done, the theory is found to predict very well the nature of the observed intermittency distributions in the LG experiments. It is demonstrated that in all the experimental runs reported there is strong evidence for the existence of a two-dimensional to one-dimensional subtransition in the transition zone.

The basic transition zone model is described in §3. The LG experiments are described in §4. In the same section, the transition zone model is supplemented to apply to an experiment where the measuring location is fixed and data are obtained at different free-stream velocities. The model is compared to the LG experiments in §5, where the data of Rao (1974) are also briefly discussed. Apart from direct comparisons between observations and the model, some information on spot propagation speeds and angles, the expected location of subtransition, as well as the implied effect of heating on these parameters, can be gleaned from the analysis. These results are presented in §6 along with some inferences on the effect of heating on the transition characteristics of the flow.

3. The subtransition model

We first review here the derivation of the expression for intermittency in axisymmetric flow.

3.1. *Intermittency in the upstream region*

In the upstream or U region, transition proceeds as in two-dimensional flow, as shown in the upstream part of figure 1. The base of the spot subtends an angle $2\alpha_s$ at its origin, where the subscript s stands for a single spot. In the absence of large

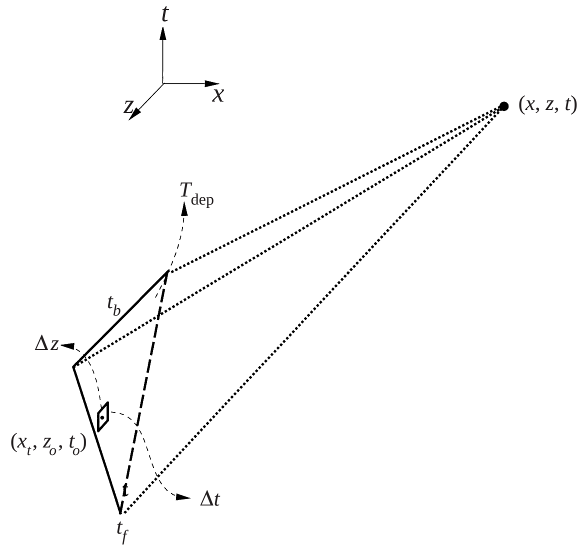

FIGURE 2. Dependence area A_{dep} at x_t within which a spot must originate in order to cause a two-dimensional spot of turbulence at (x, z, t).

pressure gradients or of strong lateral convergence/divergence of the streamlines, α_s remains constant, which means that the lateral dimension of the spot grows linearly with its distance from the point of its origin. At high rates of spot formation, several spots merge to form a spot cluster. At a certain streamwise distance $x = x^*$, the rear of a single spot (for low rates of spot formation), or the width of the spot cluster, becomes equal to the circumference of the body. Thus, if x_t is the streamwise location of transition onset (where the spots are formed), the U region is defined by $x_t \leqslant x \leqslant x^*$. We denote the coordinate along the body circumference as z and time as t, and define a function $I(x, z, t)$ which is 1 if the flow is turbulent at (x, z, t) and 0 otherwise. According to the hypothesis of concentrated breakdown, all turbulent spots originate close to a particular streamwise location x_t. They may be assumed to appear randomly along the circumference and in time, in accordance with a Poisson distribution (Narasimha 1985). With this hypothesis, the dependence volume described by Emmons (1951) shrinks to a 'dependence area' $A_{\text{dep}}(z_o, t_o)$ at x_t (where the subscript o stands for spot origin). This is an area swept out in the plane of time and spanwise coordinate, with a shape similar to that of the spot, possessing the property that $I(x, z, t)$ is unaffected by spots forming outside the area, and $I(x, z, t) = 1$ if at least one spot originates within it. A typical dependence area is shown in figure 2. For the purpose of illustration, the shape of the spot in the U regime has been approximated by an isosceles triangle in this diagram (no such specific assumption on spot shape is necessary for the model). A little consideration reveals that the base of the triangle is the same as the lateral spread of a turbulent spot at x, while the extent in the time axis is equal to the difference $t_r - t_f$ in the time of arrival at x of the front and the rear of a spot (t_f and t_r respectively). It has been widely observed that, in the absence of strong pressure gradients and at sufficiently high Reynolds numbers, the front and the rear of a spot travel at different (constant) fractions of the free-stream velocity, U. We may therefore write the proportionality relation

$$t_r - t_f \propto \frac{x - x_t}{U}. \tag{3.1}$$

For a given monitoring location (x, z), the residence time of a spot is highest for a spot originating at $z_o = z$, and decreases as z_o moves away from z on either side, in a roughly linear fashion if the spot shape is taken as a triangle (making A_{dep} also roughly triangular).

The intermittency γ is given by the joint probability of one or more spots originating in any of the elements of area $\Delta z \Delta t$ within A_{dep} shown in figure 2. This is the complement of the probability that no spot occurs within A_{dep}, which itself is the product over all area elements of the probability that no spot originates in each element. Let n_2 be the mean number of spots forming per unit time within a unit distance along the circumference at x_t. Then γ may be expressed in a form typical of a Poisson process as

$$\gamma = 1 - \exp\left[\frac{-n_2 \sigma_2 (x - x_t)^2}{U}\right], \quad x_t \leqslant x \leqslant x^*, \tag{3.2}$$

where σ_2 is a non-dimensional spot propagation parameter. This is the two-dimensional law for intermittency distribution as a function of downstream distance, as derived by Narasimha (1957).

3.2. *Intermittency in the downstream region*

At $x = x^*$, the corners of the base of the spot or spot cluster (as the case may be) touch each other for the first time, so there can be no further lateral growth. As the associated patch of turbulence continues to grow in the axial or streamwise direction, transition enters the downstream (D) stage. In the case of a single spot wrapping around, the patch of turbulence now takes the shape of a tube with a chamfered (oblique) cut at the leading end. If several spots merge before the cluster wraps itself around, the 'base' of the spot is a sleeve as before (possibly with a rather ragged trailing edge), while the 'head' is irregular in shape, as shown in figure 1.

Narasimha (1984) showed that the intermittency in the D zone follows a one-dimensional law; his argument is briefly as follows. Downstream of x^*, the spot may be taken to consist of two distinct parts: a 'head' whose shape may be a rough triangle or a possibly irregular union of triangles (depending on the spot merging scenario), and a 'base' that keeps growing (in one dimension) at the rear. The change over from two-dimensional to one-dimensional spot growth marks the subtransition within the transition zone. If γ^* is the intermittency at the location x^* where the spot just wraps itself around the body, the intermittency in the D stage may be expected to have the form

$$\gamma = \gamma^* + (1 - \gamma^*)\gamma_1, \quad x \geqslant x^*, \tag{3.3}$$

where the factor γ_1 varies from zero at $x = x^*$ to unity at $x \gg x^*$; γ^* is the (constant) probability that the flow is turbulent at x due to the presence of the head of the spot cluster, and the term $(1 - \gamma^*)\gamma_1$ is the contribution from the base.

To estimate γ_1, we note that the geometry of the dependence area at x for the base of the spot is a 'tube' $T_{\mathrm{dep}}(t_o)$, constructed by evolving over a certain period of time a ring corresponding to the circumference of the body at x_t (Fig. 3). As in the case of A_{dep} in two-dimensional flow, the axial extent of T_{dep} is given by the difference in the times of arrival at x of the front and the rear of the base of the spot, given by t_f and t_r respectively. Again, γ_1 is given by the joint probability of a spot occurring in any of the elements of area $\Delta z \Delta t$ shown in figure 3, and leads (by the same arguments as before) to the expression

$$\gamma_1 = 1 - \exp\left[-n_2 (2\pi a_t)(t_r - t_f)\right], \tag{3.4}$$

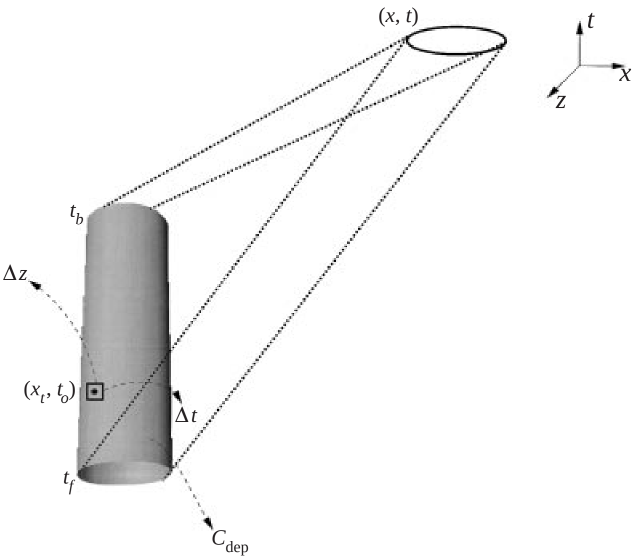

FIGURE 3. Dependence tube T_{dep} at x_t within which a spot must originate in order to cause the passage of a sleeve of turbulence at (x, t).

where a_t is the body radius at the onset of transition. In the D regime, the one-dimensional breakdown rate n_1 (i.e. the number of spots formed per unit time anywhere on the circle at x_t) is related to n_2 by

$$n_1 = 2\pi a_t n_2. \tag{3.5}$$

The propagation velocity of the turbulent spot in one-dimensional growth is found (Rao 1974) to be similar to that in the two-dimensional regime, i.e. at sufficiently high Reynolds numbers the velocities of the front and the rear of the head are constant fractions of the free-stream velocity. Since the sleeve begins to form at x^*, the difference in time of arrival of the front and rear (on average) of the base is

$$t_r - t_f \propto \frac{x - x^*}{U}. \tag{3.6}$$

Using (3.4), (3.5) and (3.6), equation (3.3) may now be written as

$$\gamma = 1 - (1 - \gamma^*) \exp\left[\frac{-n_1 \sigma_1 (x - x^*)}{U}\right], \quad x \geqslant x^*, \tag{3.7}$$

which is the one-dimensional law for the D regime proposed by Narasimha (1984). The non-dimensional spot propagation parameter σ_1 is similar to σ_2 in the U region, the two being related as follows.

Let $k_f U$ and $k_r U$ be the propagation velocities of the front and the rear of the spot respectively. Equation (3.2) has been obtained by defining

$$\sigma_2 = G \tan \alpha_s \left[\frac{1}{k_{r2}} - \frac{1}{k_{f2}}\right], \tag{3.8}$$

where G is a proportionality factor which depends on the spot geometry (for an analysis of how σ_2 and hence G can be obtained from experimental data on spots, see Narasimha 1985). In general, both σ_2 and G vary with height in the boundary layer. If we select a reference height (e.g. the one at which the spot is largest) so that all

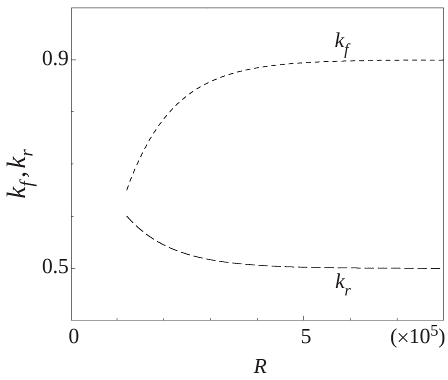

FIGURE 4. Schematic of variation with Reynolds number of the velocities of the front (k_f) and the base (k_r) of a turbulent spot.

parameters on the right of (3.8) correspond to this height, then the factor G would be 1 if the spot were perfectly triangular. In the one-dimensional regime, it is seen from equations (3.6) and (3.7) that

$$\sigma_1 \equiv \frac{1}{k_{r1}} - \frac{1}{k_{f1}}. \tag{3.9}$$

It is observed in the experiments of Rao (1974) that the factors k_f and k_r vary with Reynolds number as shown schematically in figure 4. The spot therefore grows more slowly at lower Reynolds numbers. At sufficiently high Reynolds numbers, k_{f1} and k_{r1} tend to become asymptotically constant, reaching values of about 0.9 and 0.5 respectively, the same as in two-dimensional flow. If, therefore, an axisymmetric flow experiment were to be conducted entirely in the asymptotic Reynolds number range, we could take $k_{f1} = k_{f2}$ and $k_{r1} = k_{r2}$.

A subtransition from two-dimensional to one-dimensional spot propagation on an infinitely long body may definitely be expected to occur if the circumference of the body does not increase at a rate greater than the lateral spread of one turbulent spot (Narasimha 1984). The streamwise growth of the base $2b$ of the spot is given by

$$\frac{\mathrm{d}b}{\mathrm{d}x} = \tan \alpha_s. \tag{3.10}$$

A sufficient condition for subtransition to occur and for the one-dimensional regime to be maintained downstream is that this lateral growth must be greater than or equal to the growth in the body circumference $2\pi a$, i.e.

$$\pi \frac{\mathrm{d}a}{\mathrm{d}x} \leqslant \tan \alpha_s. \tag{3.11}$$

It is shown in Appendix A that the radius of the body used in the LG experiment is slender enough for subtransition to be expected to occur on the surface, and that the streamwise increase in the radius is slow enough to be able to support a region of one-dimensional spot growth downstream of the location of subtransition. Note that relation (3.11) is not a necessary condition for subtransition to occur. If several spots merge with each other and the resulting cluster can wrap itself around the body, subtransition to one-dimensional spot growth may occur even when (3.11) is not satisfied, as discussed in §6. It is shown in that section that the expected location of subtransition may be determined from the transition onset Reynolds number and the breakdown rate.

4. Intermittency variation with Reynolds number at fixed station

LG performed transition experiments in a water tunnel on a heated axisymmetric body whose axis is aligned with the flow direction. The body shape is described mathematically by a modified ellipse function defined in detail by LG. The pressure gradient over the body is mildly favourable. The pressure gradient parameter β, given by

$$\beta = \frac{2m}{m + 1 + m_a} \tag{4.1}$$

where

$$m = \frac{x}{u_e}\frac{\mathrm{d}u_e}{\mathrm{d}x} \quad \text{and} \quad m_a = \frac{2x}{a}\frac{\mathrm{d}a}{\mathrm{d}x}, \tag{4.2}$$

and u_e is the velocity at the edge of the boundary layer, has been shown by LG to have a practically constant value around 0.04 except at the nose. The intensity of free-stream turbulence is constant at around 0.1% of the free-stream velocity throughout most of the velocity range used in the heated-body experiments. Measurements of intermittency and burst rate are made at a fixed station on the body corresponding to an arclength $s = 2.12\,\mathrm{m}$ from the nose of the body. Different Reynolds numbers are achieved by varying the free-stream velocity, and the intermittency is obtained as a function of the Reynolds number

$$R \equiv Us/\nu, \tag{4.3}$$

where U is the free-stream velocity at a reference location and ν is the kinematic viscosity.

There is a fundamental difference between this method of measuring intermittency and the more common procedure of maintaining a constant reference free-stream velocity and measuring the intermittency as a function of the downstream distance. Equations (3.2) and (3.7) are valid only for the latter kind of experiment. When the external velocity is not maintained constant, the location of transition onset x_t, the factors $n_2\sigma_2$ in equation (3.2) and $n_1\sigma_1$ in equation (3.7), the location of subtransition (x^*), and the intermittency γ^* at subtransition cannot be taken as constant. The equations therefore require careful recasting before the model can be tested against the LG experiments. (The data of Rao 1974, on the other hand, can be used directly to check the model as shown by Narasimha 1984.)

We first examine the U zone of transition in the LG experiments. In the flow over an unheated two-dimensional body, for a given pressure gradient and free-stream turbulence level, the transition onset Reynolds number is approximately constant and can be obtained, for example, from an empirical correlation of the type given by Govindarajan & Narasimha (1991). In the LG experiment, we have already seen that both the free-stream turbulence and the pressure gradient are for all practical purposes constant in the region of interest. Furthermore, since the heating level is maintained constant during the course of any one experiment, the transition Reynolds number may be assumed to take on a constant value for each level of heating, i.e. for each set of intermittency data.

Next we consider the transition zone length λ. In the U (or two-dimensional) zone, this length is defined as the distance between the (possibly hypothetical) stations where γ would equal 0.25 and 0.75 respectively if the spot growth were to continue to be two-dimensional throughout. Using (3.2), the length λ may be expressed as

$$\lambda^2 = 0.411\frac{U}{n_2\sigma_2}, \tag{4.4}$$

so a specification of λ may more fundamentally be seen as a substitute for the spot parameter product $n_2\sigma_2$.

Now it has been shown (Abu-Ghannam & Shaw 1980; Narasimha 1985) that, for a given pressure gradient and free-stream turbulence level, the value of R_λ depends only on R_t to a first approximation. Thus, if R_t is taken as constant it is logical to take R_λ also as a constant for each heating level. This is consistent with the correlation

$$R_\lambda = 9R_t^{3/4} \tag{4.5}$$

proposed by Narasimha (1984) on the basis of a compilation of all available data on a flat plate. The validity of (4.5) for two-dimensional spot growth has been established by Dey & Narasimha (1991) who showed that a non-dimensional spot formation parameter based on (4.5) is practically constant across a wide range of experiments at a given pressure gradient and free-stream disturbance level. The transition zone length predicted by the above correlation is compared to that of Chen & Thyson (1971) in Appendix B. The concept of a constant spot formation parameter for a given pressure gradient has been since used by several workers (e.g. Walker 1998; Johnson 1998) and found to give reasonable transition predictions. Correlations for the dependence of the spot propagation parameter on pressure gradient and on free-stream turbulence may be found in Gostelow, Melwani & Walker (1995).

Using (4.4) and (4.5), it is easily seen that $n_2\sigma_2$ scales with the cube of the Reynolds number, as given by

$$\frac{n_2\sigma_2}{R^3} \propto \frac{\nu}{s^3 R_t^{3/2}}. \tag{4.6}$$

The non-dimensional quantity N_2, defined (in analogy with the parameter N in Narasimha 1985) as

$$N_2 \equiv \frac{n_2\sigma_2 s^3}{R^3\nu}, \tag{4.7}$$

may thus be expected to be constant for a given heating level.

Equation (3.2) may now be rewritten in terms of R as follows:

$$\gamma = 1 - \exp\left[-N_2(R-R_t)^2\right], \tag{4.8}$$

or

$$F_2 \equiv [-\ln(1-\gamma)]^{1/2} = \sqrt{N_2}(R-R_t). \tag{4.9}$$

A plot of F_2 versus R should therefore yield a straight line in the U regime.

The constancy of R_t may be used to formulate an intermittency model in the D regime as well. Equation (3.7) may first be rewritten in terms of Reynolds numbers as

$$\gamma = 1 - (1-\gamma^*)\exp\left[\frac{-n_1\sigma_1(R-R^*)s^2}{R^2\nu}\right], \quad x \geqslant x^*. \tag{4.10}$$

Here, from equation (3.5), and using the fact that σ_1 is related to σ_2 by the geometry of spot propagation as described by equations (3.8) and (3.9), it is immediately evident that $n_1\sigma_1$ too scales with R^3. Analogously with the definition (4.7) of N_2 in the U zone, a non-dimensional spot formation parameter is defined for the D zone as

$$N_1 \equiv \frac{n_1\sigma_1 s^3}{R^3\nu a_t}, \tag{4.11}$$

which again may be assumed to remain constant for a given heating level.

The Reynolds number of subtransition R^* will be shown in § 6 to be constant for a

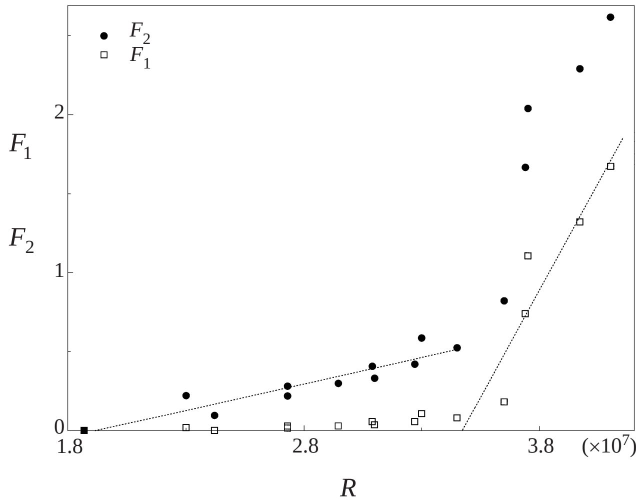

FIGURE 5. Functions F_2 and F_1 given by equations (4.9) and (4.13) respectively, for the data set LG4. The point of subtransition may clearly be observed. Dashed lines: best fits.

given heating level. We make use of this result here, and write equation (4.10) in the form

$$\gamma = 1 - (1-\gamma^*)\exp\left[-\frac{N_1 a_t}{s}(R-R^*)R\right] \tag{4.12}$$

or

$$F_1 \equiv -\frac{1}{R}\ln\left[\frac{(1-\gamma)}{(1-\gamma^*)}\right] = \left[\frac{N_1 a_t}{s}(R-R^*)\right], \tag{4.13}$$

which again shows a linear relationship between F_1 and R.

The expressions (4.9) and (4.13) are analogues of the straight line relationships for experiments where the free-stream velocity is maintained constant, derived respectively for the two-dimensional and one-dimensional spot growth regimes by Narasimha (1957, 1984). These relations are devised to enable quick checks of the expressions against experimental data and to obtain estimates of transition-zone parameters.

5. Comparison with experiment

The quantities F_2 and F_1 defined in equations (4.9) and (4.13) can be computed for the intermittency data of LG and plotted against R; an example is shown in figure 5. It is observed that the experimental points fall on the respective straight lines in the two-dimensional and one-dimensional regimes (low and high γ respectively). In addition, the Reynolds number at which the switch-over from two-dimensional to one-dimensional spot growth occurs can be obtained with good accuracy from these plots.

The data at each heating level have been analysed by this method, and the slopes and intercepts of the best fitting straight lines in the F_1, R and F_2, R plots are obtained. The intermittency distributions calculated using equations (4.8) and (4.12) are shown plotted along with the experimental data in figure 6. From all the plots in figure 6, it is clear that the variation of the intermittency with streamwise distance is qualitatively very different in the downstream part of the transition zone from that in the upstream region. In addition, the increase in γ is gradual in the U region and rapid in the D region. These are the ‘footprints’ of the subtransition discussed earlier. The excellent

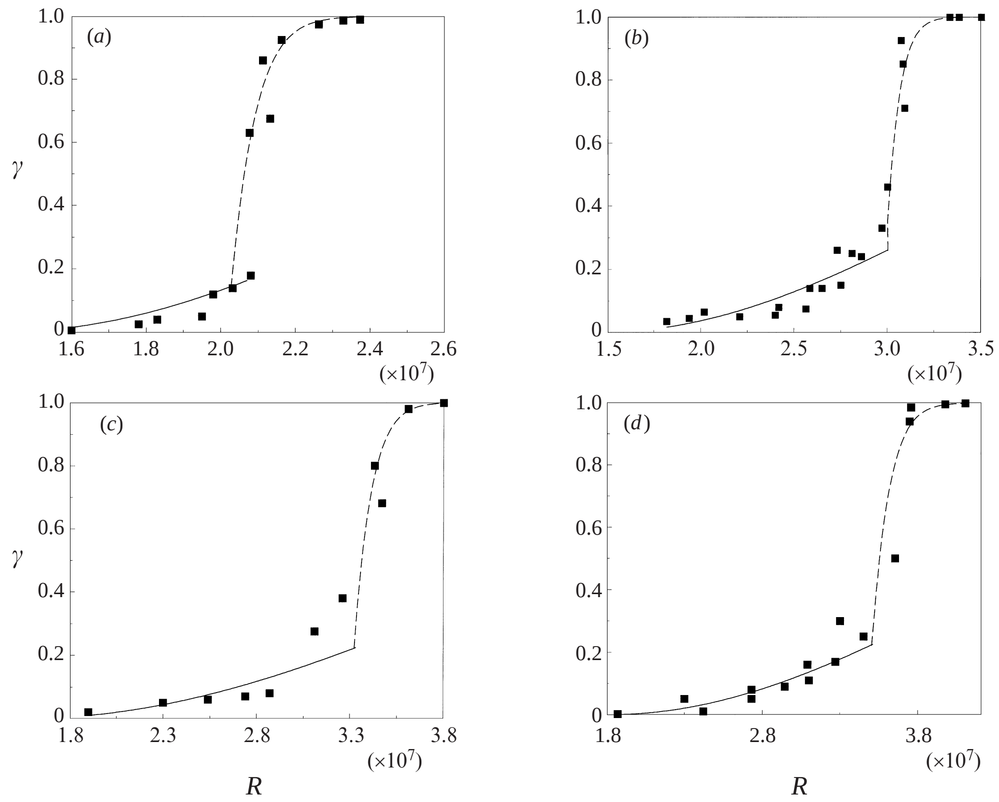

FIGURE 6. Comparison of present model with the experiments of LG. Solid line: modified two-dimensional law; dashed line: modified one-dimensional law; symbols: experiment. The data sets are (*a*) LG1; (*b*) LG2; (*c*) LG3; (*d*) LG4, as tabulated in table 1.

agreement between experiment and model seen in these diagrams gives us confidence in drawing conclusions on the effect of heating on the transition zone.

The experiments of Rao (1974) have been conducted in the more traditional manner, i.e. intermittency is measured as a function of downstream distance for a given free-stream velocity. While he recognized that spot propagation on an axisymmetric body is two-dimensional upstream and one-dimensional downstream, he did not propose a model for spot propagation that could explain his observations in the one-dimensional regime. As shown by Narasimha (1984) the original equations (3.3) and (3.9) can be used to predict the intermittency distributions in his experiments. The desired parameters at the onset of transition and subtransition respectively can be obtained from the following straight-line relationships corresponding to (3.3) and (3.9) respectively (Narasimha 1984):

$$G_2 \equiv [-\ln(1-\gamma)]^{1/2} = \sqrt{\frac{n_2\sigma_2}{U}}(x - x_t), \tag{5.1}$$

and

$$G_1 \equiv \ln\left[\frac{1-\gamma}{1-\gamma^*}\right] = -\frac{n_1\sigma_1}{U}(x - x^*). \tag{5.2}$$

The intermittency data of Rao (1974) for two cylinders of different radii, each at two

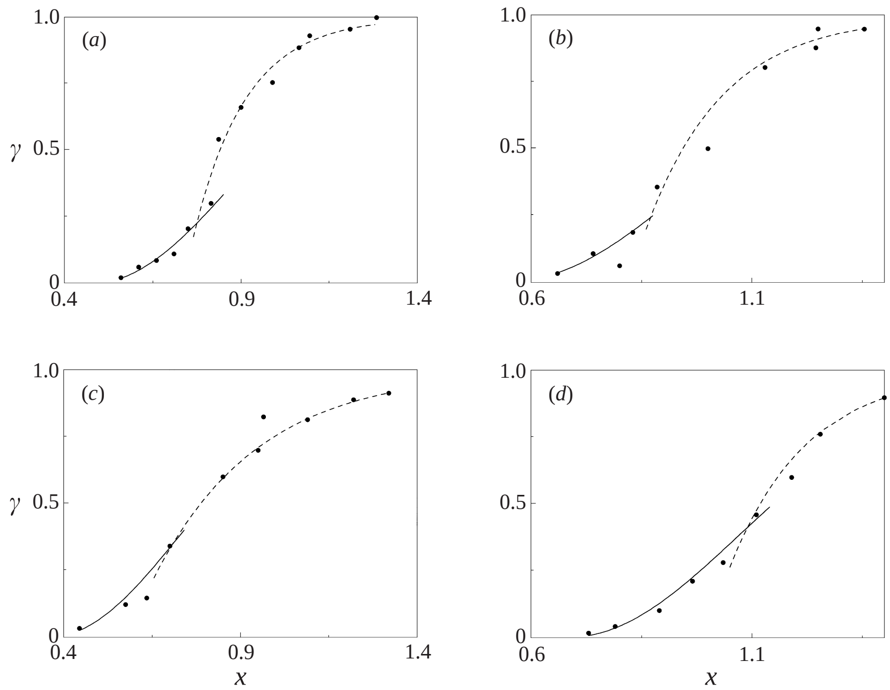

FIGURE 7. Comparison of present model with the experiments of Rao. Solid line: two-dimensional law; dashed line: one-dimensional law; symbols: experiment. (*a*) R1; (*b*) R2; (*c*) R3; (*d*) R4, as tabulated in table 2.

Reynolds numbers (R_a based on the radius of the body) are compared with present calculations in figure 7. Again, the model is seen to perform well.

6. Discussion

With the very satisfactory agreement of the experimental data with the subtransition model for intermittency on an axisymmetric body, we are now in a position to examine the implications of the present analysis for the effect of heating on transition.

6.1. *Transition onset*

Now LG use the Reynolds number at which the intermittency is 0.5, say $R_{0.5}$, as a measure of the location of transition. They show that at low levels of heating, transition is delayed by heating, i.e. $R_{0.5}$ increases with heat input, while at high heating there is little further increase. They present a qualitative comparison between the instability Reynolds number R_{cr} and $R_{0.5}$ in this range, and conclude that the latter saturates while the former does not. They attribute the saturation in $R_{0.5}$ to such factors as free-stream disturbances, surface flaws, flow asymmetries and system idiosyncracies which may be driving transition at the higher heating levels. As we have already pointed out, the free-stream turbulence is approximately constant for the entire range of the LG experiments on the heated body, and, from the extensive evidence available on the effect of free-stream turbulence, is unlikely to be a factor in accounting for the observed changes in onset Reynolds numbers at different heating levels.

We propose here that the quantity more suited for comparison with the trends of the instability Reynolds number R_{cr} is R_t, the Reynolds number at which the flow

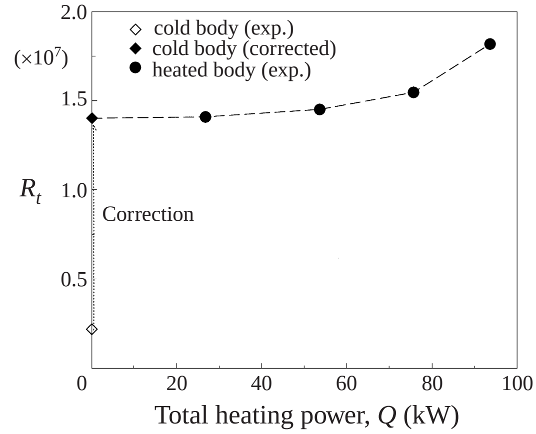

FIGURE 8. Variation of the transition onset Reynolds number with level of heat.

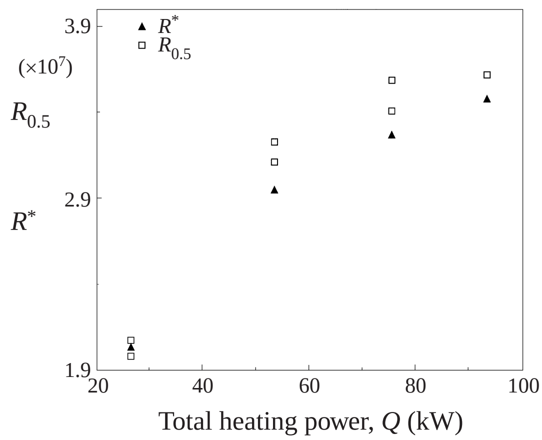

FIGURE 9. Variation of the two-dimensional to one-dimensional subtransition Reynolds number R^*, and the LG Reynolds number $R_{0.5}$ with level of heat.

breaks down and the first turbulent spots appear. The Reynolds number at 50% intermittency, being downstream of subtransition, is less directly related to R_{cr} as discussed in detail below.

The four heated flow experiments of LG are denoted by LG1 to LG4 in increasing order of heating. The Reynolds numbers R_t at transition may be estimated from the intercept of the best-fit line for F_2 according to (4.9), while the Reynolds number R^* at subtransition in each experiment is the location at which the two-dimensional and one-dimensional curves intersect in figure 6. The transition Reynolds number R_t is plotted against heating level in figure 8, while the subtransition Reynolds number and the LG measure $R_{0.5}$ are similarly plotted in figure 9.

Before discussing the heated cases at length, it is relevant to examine the experimental result for transition over the cold body. The range of R across the transition zone over the cold body is approximately 0.2×10^7 to 0.5×10^7, which corresponds to free-stream velocities of 1 to $2.5\,\mathrm{m\,s^{-1}}$ in the water tunnel. The lowest velocity of operation in the heated case, however, is about $7.5\,\mathrm{m\,s^{-1}}$. The large difference in tunnel operation velocity has a significant impact on the transition onset, since the free-stream disturbance level, q, is very different in the cold and in the heated flow experiments. From figure 1 of LG it is observed that above a free-stream velocity of about $6.5\,\mathrm{m\,s^{-1}}$, q settles down to a value of about 0.1% nearly independently of tunnel speed. At low speeds, however, q is much higher – the value corresponding to the estimated R_t for the cold body, though not shown explicitly in the LG paper, appears to be of the order of 0.5%. This large increase in q would cause a significant decrease in the transition onset Reynolds number. This is evident upon examining the empirical correlation for transition onset (Govindarajan & Narasimha 1991) in the absence of pressure gradient:

$$R_{\theta t} = 110 + \frac{340}{(q^2 + q_0^2)^{1/2}}. \tag{6.1}$$

Here, the onset Reynolds number is based on the momentum thickness θ of the boundary layer, and q_0 stands for the residual non-turbulent disturbances in the tunnel.

The relation (6.1) may be used to obtain a corrected value of R_t for the cold body. This corrected value R_t' is the Reynolds number at which transition onset would be expected to occur on the cold body if the free-stream turbulence had been maintained at the same value (0.1%) as in the heated flow experiments. Since experimental data on noise levels in the LG tunnel are not available, we use here a typical value of 0.1%. With $q = 0.47\%$, transition onset according to (6.1) occurs at the value reported by LG. For $q = 0.1\%$, on the other hand, we get $R_t' \simeq 1.4 \times 10^7$. This corrected value for the cold body must be taken as very tentative, but, when plotted in figure 8, it is seen to be consistent with the trend followed by R_t in the heated flow experiments. Since the intermittency data on the cold body are very sparse (only three points within the transition zone), the corresponding intermittency distribution has not been modelled.

It is immediately apparent from figures 8 and 9 that the behaviour of the transition Reynolds number R_t is different qualitatively from that of $R_{0.5}$, in that it actually increases at a faster rate at higher heating levels than at lower. From the values of instability Reynolds number R_{cr} computed by Wazzan *et al.* (1968) we find that, for the range of overheat in which the LG experiments have been conducted (below 30°C), the critical Reynolds number based on the *boundary layer thickness* goes up linearly with level of overheat. This means that the critical Reynolds number based on the *streamwise coordinate* scales roughly as the square of the overheat level, i.e. goes up faster at higher levels of heating. The behaviour of R_t shown in figure 8 is consistent with this result.

The Reynolds number R^* at which spot-wrapping occurs depends on several other parameters besides R_t, such as the ratio of the lateral spread to the axial extent of the spot. It is seen from figure 9 that the variation of $R_{0.5}$ is similar to that of R^*, which is to be expected, since it lies in the one-dimensional spot growth regime.

From the viewpoint of stability theory, it is more meaningful to plot the variation of transition Reynolds number with the wall overheat temperature ΔT rather than with the total heat input. The overheat has been measured by LG for only one velocity per heating level. The variation of R_t, R^* and $R_{0.5}$ with this ΔT is shown in figure 10. The transition-onset Reynolds number still goes up more rapidly at higher levels of heat. It is interesting to note, however, that the LG measure of $R_{0.5}$ does not appear to saturate any more, although it still does not increase as rapidly as R_{cr}.

A detailed study of the relationship between transition and stability in a heated underwater body will be presented elsewhere. However, it is relevant to mention here that in the work of Wazzan *et al.* (1968) the instability critical Reynolds number

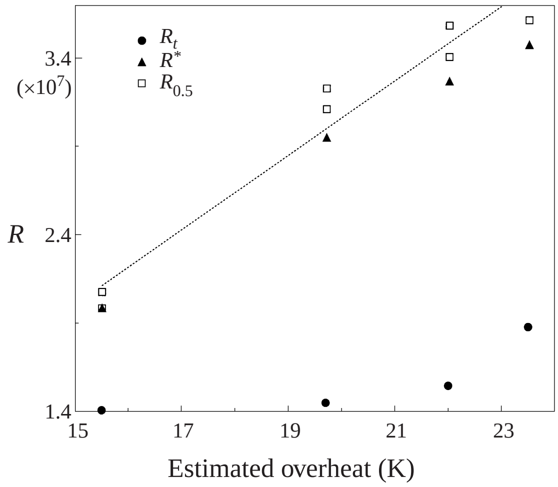

FIGURE 10. Variation of R_t, R^* and $R_{0.5}$ with estimated degree of overheat.

R_{cr} for a heated flat plate goes up much more rapidly than the transition Reynolds number in the LG experiment seems to do. This is similar to the behaviour of R_{cr} and R_t for two-dimensional accelerated flow. In the case of highly accelerated flow, at moderate levels of free-stream turbulence, R_t can often be much lower than R_{cr} (as is evident from the data available in Abu-Ghannam & Shaw 1980, for example). A comparison of estimated R_t in the LG experiments with the heated flat-plate results (R_{cr}) of Wazzan *et al.* shows them to be qualitatively the same.

6.2. *Spot propagation parameters*

The streamwise location x^* of subtransition and the relative magnitudes of σ_1 and σ_2 have implications for the geometry of spot growth in the two-dimensional regime, and the speed of spot propagation in the two regimes, which are discussed below.

From the experimental data, we can surmise whether subtransition occurs due to a single spot wrapping itself around the body or a cluster of spots doing so. The first scenario is likely to occur when the time T_s taken by a single spot to wrap itself around the body is much less than the average time interval T_m between the formation of two successive spots anywhere around the circumference of the body at x_t. The two times are given respectively by

$$T_s = \frac{x^* - x_t}{k_r U}, \quad T_m = \frac{1}{2\pi a_t n_2}. \tag{6.2}$$

We now define a non-dimensional 'merge' parameter

$$M \equiv \frac{T_s}{T_m} = \frac{2\pi^2}{k_r \tan \alpha_s} \frac{n_2 a_t^2}{U}, \tag{6.3}$$

which provides a criterion for the single (low M) or cluster (high M) scenario. If the boundary layer is close to Blasius, the above expression may be rewritten (see Appendix C) as

$$M \simeq 3R_t^{1/2} \frac{a_t^2}{x_t^2}. \tag{6.4}$$

As M increases beyond 1, it is more probable that one spot will merge with another before either (singly) wraps around the body. The combined width of the spots is

Exp.	Heat (kW)	$R_t(\times 10^{-7})$	$R^*(\times 10^{-7})$	M	α_{eff}(deg.)	$K \equiv (G \tan \alpha_s)\sigma_1/\sigma_2$
LG1	26.6	1.407	2.037	124	37.2	1.7
LG2	53.4	1.450	3.002	262	24.4	6.2
LG3	75.4	1.547	3.322	292	23.7	7.8
LG4	93.3	1.819	3.503	247	26.0	4.7

TABLE 1. Parameters derived from the LG experiment.

Exp.	Radius (m)	R_a	x_t (m)	x^* (m)	γ^*	M	α_s (deg.)	$K \equiv (G \tan \alpha_s)\sigma_1/\sigma_2$
R1	0.0095	13000	0.480	0.775	0.227	0.95	5.8	3.8
R2	0.0095	6450	0.547	0.870	0.242	0.55	5.3	3.3
R3	0.025	22000	0.357	0.670	0.246	8.24	14.1	1.9
R4	0.025	6300	0.686	1.126	0.469	1.66	10.1	2.3

TABLE 2. Parameters derived from the Rao experiment.

given by the union of the widths occupied by each spot. If a spot merges just before it wraps, i.e. if M is just slightly greater than 1, the contribution of the second spot is small, and subtransition is basically the same as that achieved by a single spot. For wrapping to occur with a cluster of spots, we must have $M \gg 1$. The two-dimensional regime may therefore be expected to shrink as M increases.

For each LG experiment, the intermittency data may be used along with equation (6.4) to obtain the merge parameter M. The values so obtained are listed in table 1. The corresponding values for the experiments of Rao are listed in table 2. It is immediately obvious that the wrapped spot must be single in Rao's experiments and clustered in the LG experiments. Subtransition may therefore be expected to occur at a much shorter distance in the LG experiments than in the Rao experiments. To verify this prediction, we examine the effective half-angle α_{eff} subtended by the patch of turbulence at its origin, defined by

$$\tan \alpha_{\text{eff}} = \frac{x^* - x_t}{\pi a}. \tag{6.5}$$

It is seen that while α_{eff} for Rao's data is not far from 10°, the value in the LG experiments is always substantially higher, suggesting that the wrap scenario corresponds to a spot cluster. The data set R3 shows a somewhat larger M as well as a larger α_{eff} than the other three sets of Rao's data, which is consistent with the ideas proposed here.

We now look at the ratio of spot propagation parameters in the two-dimensional and one-dimensional regimes, obtained from equations (4.7), (4.11) and (3.5):

$$\frac{\sigma_1}{\sigma_2} = \frac{N_1}{2\pi N_2}. \tag{6.6}$$

On comparing the expressions for σ_2 and σ_1 given by (3.8) and (3.9) respectively, it may be seen that for a quantitative comparison, it is convenient to examine the parameter $K \equiv (G \tan \alpha_s)\sigma_1/\sigma_2$. (As mentioned before, α_s is the half-angle subtended by a *single* spot at its origin.) In view of the above arguments, in the Rao experiment we may directly use the α_s estimated from the data; while for the LG data, in the absence of direct measurements of α_s, we use the value of 10° which has been observed in a variety of experiments to be the half-angle subtended by a single spot. The factor

G depends to some extent on the deviation of the two-dimensional spot from an exactly triangular shape, but this effect is much smaller than that of the height at which measurements are made. In the two-dimensional regime, the spot exhibits a long overhang, and appears very different at different heights away from the wall (Gad-el-Hak, Blackwelder & Riley 1981); from the available data we estimate that, at the wall, the streamwise extent of the spot is half of its maximum value. If it is assumed (in the absence of quantitative information about the variation of the lateral extent) that the spot width at the wall is again about half the maximum width, G at the wall will be about 1/4 of the value at its maximum. Far downstream in the one-dimensional growth regime the overhang effect (accounted for in the 'head' of the spot) is negligible, so G varies little.

In Rao's experiments, the measurements (using a hot wire) were made close to the maximum extent of the spot, so G may be taken to be about 1; whereas in the LG experiments (where the probes are flush-mounted hot films, and measure intermittency at the wall), G may be taken to be 1/4. The ratio K for the two experiments is shown in tables 1 and 2. The values are all of comparable order of magnitude, confirming the general validity of the present discussion. However, except for the data set LG1, the value of K for the LG results is noticeably higher than for Rao's results. It is possible that heating the wall has the effect of making the spot smaller (as would be expected for a stabilizing influence), in which case the factor G would be reduced, and the ratio K would become closer to the Rao results.

Since spot merging and wrapping occur in the two-dimensional growth regime, we may expect that the only quantities which determine the merging process will be the transition onset Reynolds number and the spot formation parameter. This is shown to be the case by detailed statistical arguments in Appendix D. In fact, the location of subtransition may be determined from these arguments. The intermittency γ^* has been shown (according to equation (D 10)) to be constant for a given heating level. For the LG experiment, this expression gives

$$R^* = R_t + \left(\frac{G}{N_2}\right)^{1/2}, \qquad (6.7)$$

i.e. for a given heating level, R^* is constant and depends only on the transition onset Reynolds number and the spot formation rate. The effective spread angle α_{eff} may now be estimated, since equation (6.7) may be written as

$$\tan\alpha_{\mathrm{eff}} = \frac{\pi a}{s}\left[\frac{N_2^{1/2} R_t}{G^{1/2}} + 1\right]. \qquad (6.8)$$

It is shown in figure 11 that, with $G = 1/4$, the spread angle predicted by equation (6.8) is close to that obtained in the analysis of the LG results. Only the first experiment, LG1, does not fit in as well with the other results – in this case a value $G = 0.16$ agrees better with observations. The reasons for this are not clear.

It must however be remembered that the values of spread angle and spot propagation ratio shown in the tables have been deduced indirectly from the experimental data. It is therefore desirable that the present deductions on the effect of surface heating on spot propagation characteristics are checked with detailed experiments in which measurements are preferably made at successive streamwise locations while maintaining the tunnel speed constant. This approach has the advantage that the modelling is more direct and that streamwise variations in temperature do not change transition onset.

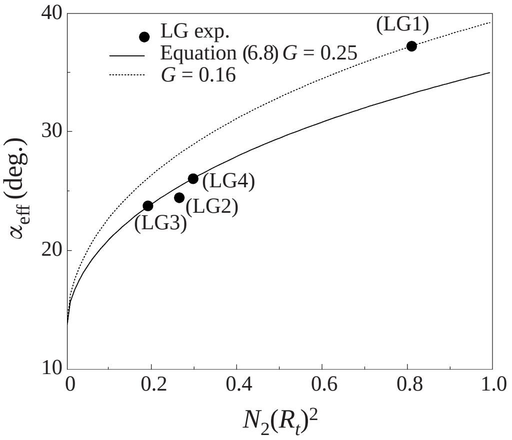

FIGURE 11. Comparison of estimated spread-angle with that derived from the LG experiment.

7. Conclusions

The subtransition model of Narasimha (1984) has been extended to the case where intermittency measurements are made at a fixed station at varying velocities over an axisymmetric body. Strong evidence for two-dimensional to one-dimensional subtransition within the transition region is found in the experimental results of LG. The assumptions of constant R_t and R_λ for a given level of free-stream turbulence, pressure gradient, tunnel disturbance and heating level are borne out by the close agreement of the model with experiment.

It is inferred from the present work that the Reynolds number $R_{0.5}$ used by LG to indicate transition onset lies in the one-dimensional spot propagation regime. This Reynolds number increases with heat at low levels of heating, but appears to saturate beyond some heating level. The present work however enables us to examine the effect of heat on the true transition onset Reynolds number R_t. As R_t is the Reynolds number at which the first breakdowns occur, it may be expected to be more directly related to the instability critical Reynolds number and is therefore a better indicator of the effect of heat on delay in onset. In the intermittency model presented here, R_t may be obtained in a straightforward manner, and is shown to increase approximately quadratically with the heating, which is consistent with the prediction of linear stability computations.

Spot growth characteristics have been indirectly obtained from the intermittency distributions. It is inferred that the spot wrapping scenario is different in the LG and Rao experiments. In the former, several spots merge into a cluster which wraps itself around the circumference well upstream of the location where a single spot would, resulting in earlier subtransition. In Rao's experiments, the scenario is of a single spot wrap–around. Direct measurements of propagation rates and spread angles on heated axisymmetric bodies (with carefully controlled temperature) are needed to confirm these predictions.

The derived spot propagation ratio K for the LG experiment is somewhat higher than in the Rao experiments, suggesting that heating reduces the two-dimensional spot size at the wall, as may be expected from a stabilizing agency.

On the whole, a remarkably consistent picture of the effect of heating on transition delay has emerged from the present work. It is paradoxical that, according to the present work, the delay in transition onset with heating is appreciable, as suggested by stability theory, but the anticipated gains are likely to be reduced because of the shrinking of the transition zone, which should be particularly severe at high Reynolds numbers because the wrapping scenario on axisymmetric bodies is then dominated by spot clusters, not single spots.

We are studying the effect of curvature and heat on streamwise and crossflow instabilities on spatially developing axially symmetric boundary layers, which will be presented elsewhere.

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Appendix A. Check for occurrence of subtransition on LG body

The LG body shape is described by a modified ellipse function, with the surface obeying the equation

$$\hat{a} = [\hat{x}(2-\hat{x})]^{1/2} - \frac{l_n}{D_{max}}\frac{C_0\hat{x}}{2p^2}\exp\left\{-\frac{\hat{x}^2}{2p^2} - k\frac{l_n}{D_{max}}\right\} + \hat{x}^2\epsilon_L, \tag{A 1}$$

where

$$\hat{x} = \frac{x}{l_n}, \quad \hat{a} = \frac{a}{\frac{1}{2}D_{max}}, \quad \text{and} \quad \epsilon_L = \frac{l_n C_0}{2p^2 D_{max}}\exp\left\{-\frac{1}{2p^2} - \frac{kl_n}{D_{max}}\right\}. \tag{A 2}$$

Here x is the axial coordinate and a is the body radius. For the body under consideration, $l_n = 2.44$ m, $D_{max} = 0.32$ m, $p = 0.3$, $C_0 = 0.0303$ and $k = 0.45227$.

The quantity $\pi \mathrm{d}a/\mathrm{d}x$ may be computed as a function of x using (A 1) and (A 2), and the angle $\delta = \tan^{-1}(\pi \mathrm{d}a/\mathrm{d}x)$ is plotted in degrees as a function of x in figure 12. It is seen that at the location of subtransition and everywhere downstream, $\delta < \alpha_s$ and the condition (3.11) is satisfied for all the experiments. The body is thus cylinder-like in the characteristics of its transition zone, for our purposes, i.e. we may expect the spot growth to undergo a subtransition from two-dimensional to one-dimensional, and downstream of this subtransition the spot growth will continue to be one-dimensional.

Appendix B

Using equation (4.4), we may rewrite equation (3.2) as

$$\gamma = 1 - \exp\left[-0.411(x-x_t)^2\right], \quad x_t \leqslant x \leqslant x^*. \tag{B 1}$$

Chen & Thyson (1971, CT) prescribe the following correlation for the transition zone length λ_{CT}:

$$R_{CT} = 60R_t^{2/3}. \tag{B 2}$$

Here, $R_{CT} = \lambda_{CT}U/\nu$, where λ_{CT} is the distance between the onset x_t of transition and the location where the intermittency is 0.95. If the spot growth is always two-dimensional, the relationship between λ_{CT} and λ may be obtained from equation (B 1) for a constant-velocity experiment as

$$\log(1-0.95) = 0.411\frac{\lambda_{CT}^2}{\lambda^2}, \quad \text{or} \quad R_{CT} = 2.7R_\lambda. \tag{B 3}$$

The correlation of CT may now be rewritten as

$$R_\lambda = 22.2R_t^{2/3}. \tag{B 4}$$

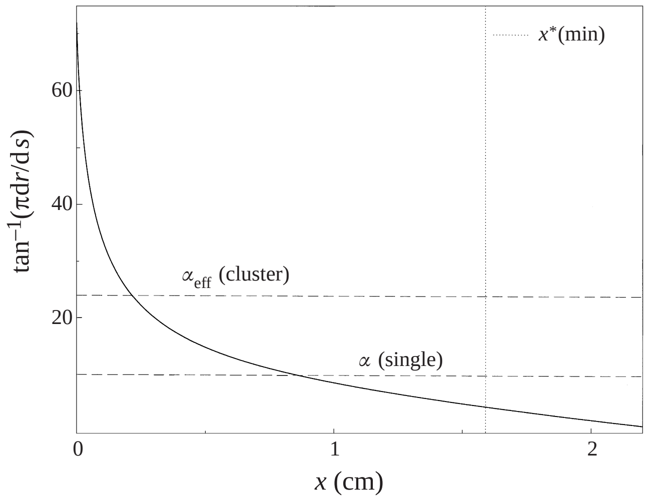

FIGURE 12. Downstream growth in the body radius. It is possible for a one-dimensional growth regime to be supported on the body if the solid line is below the dashed line.

The transition zone length predicted by the above correlation is about 5% smaller than that obtained from equation (4.5). Since the locations corresponding to $\gamma = 0.25$ and $\gamma = 0.75$ can be measured far more accurately than the locations at which $\gamma = 0$ and $\gamma = 0.95$, a correlation employing λ rather than λ_{CT} is likely to be the more reliable.

The CT transition model does not take into account any possible subtransition to one-dimensional spot growth, and is therefore valid only when spot-wrapping does not occur.

Appendix C

The spot formation rate n_2 may be deduced from the non-dimensional spot propagation parameter N (Narasimha 1985) given by

$$N \equiv \frac{n_2 \sigma_2 \theta_t^3}{\nu}, \tag{C 1}$$

where θ_t is the momentum thickness at transition. For a Blasius boundary layer,

$$\frac{\theta_t^3}{\nu} = \frac{(0.664)^3 x_t^2}{U R_t^{1/2}}. \tag{C 2}$$

In a variety of flow situations at zero pressure gradient, at not too small free-stream turbulence levels, N has been shown to have a numerical value of about 0.7×10^{-3}. Using this number, and the relation (3.8) with $\alpha_s = 10^\circ$, $k_r = 0.5$, $k_f = 0.9$ and $G = 1$, we get

$$M \sim R_t^{1/2} \frac{a_t^2}{x_t^2}, \tag{C 3}$$

with a proportionality constant close to 3.

Appendix D. Estimate of location of subtransition

Consider a single turbulent spot whose base is located at x. We label this spot S_o. If there were no other spots around the circumference at x, the width of the patch of turbulence at x would be equal to the base of the spot S_o, denoted by $2b$. The object of the discussion below is to estimate the total extra width W occupied by other spots at x. When the sum $2b+W$ equals the circumference of the body, the patch of turbulence would have wrapped itself around the body and the growth downstream would be one-dimensional.

It is necessary, therefore, to find the *average* additional portion of the circumference at x being occupied by turbulence, given that $2b$ is already occupied by the first spot, S_o. If the present time is taken to be t, the spot S_o was formed at time $t-(x-x_t)/k_{r2}U$. Any other spot forming during the time interval $\Delta t = (x-x_t)/k_{r2}U - (x-x_t)/k_{f2}U$ will also occupy some part of the circumference at x at time t.

At the location x_t where spots form, consider the circumference of the body to be divided into infinitesimal elements, each of length ΔC. The probability that a new spot S_n has originated during the time interval Δt in a particular element located l elements away from the centre of the original spot is given by

$$p = n_2 \Delta C \Delta t, \tag{D 1}$$

which can be rewritten using equation (3.8) as

$$p = \frac{n_2 \sigma_2 (x-x_t)}{G \tan \alpha_s U} \Delta C \equiv Z \Delta C, \tag{D 2}$$

which defines the quantity Z. If $l\Delta C < 2b$, the new width contributed by spot S_n will be between 0 and $l\Delta C$, depending on its time of formation. The average new width contributed by such a spot may therefore be taken as

$$w = \frac{l \Delta C}{2}. \tag{D 3}$$

The quantity w constitutes the new width of turbulence contributed by S_n provided no other spot originates in the same time interval between S_o and S_n. For small p, the probability q of no other spot forming in between is approximately

$$q = 1 - (l-1) p \Delta C. \tag{D 4}$$

A spot originating at a distance greater than $2b$ will contribute an average new width covered by turbulence of b. The total new width W given one spot may now be written as

$$W = 2 \sum_{l=1}^{2b/\Delta C} pwq + \sum_{l=2b/\Delta C}^{(2\pi a - 2b)/\Delta C} pbq, \tag{D 5}$$

where the factor 2 in the first term is present to account for both sides of S_o. Since $2b$ is already covered by S_o, the second sum in the above expression should be set to zero if $2\pi a \leqslant 4b$. Substituting (D 2) to (D 4) in (D 5), and summing the series, we get

$$W = 2Z\,[\pi a - b]\, b, \quad 2b \leqslant \pi a, \tag{D 6}$$

$$= 2Zb^2, \quad 2b > \pi a. \tag{D 7}$$

The patch of turbulence will wrap itself around the body if

$$2b + W \geqslant 2\pi a. \tag{D 8}$$

In all the LG experiments, $2b \leqslant \pi a$ at subtransition. For this case, condition (D 8) simplifies to

$$Zb \geqslant 1, \quad 2b \leqslant \pi a. \tag{D 9}$$

Using equation (3.2), it is seen that subtransition may be expected to occur if

$$\gamma \geqslant 1 - \mathrm{e}^{-G} \quad \text{while} \quad 2b \leqslant \pi a. \tag{D 10}$$

At low rates of spot formation, the width $2b$ of a single spot may exceed half the circumference before condition (D 10) is satisfied.

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